11.3 The Binomial Theorem

Investigate Patterns in Pascal's Triangle

In 1653, Blaise Pascal, a French mathematician, described a triangular array of numbers corresponding to the number of ways to choose *r* elements from a set of *n* objects. Some interesting number patterns occur in Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Fill in the next row of Pascal's triangle. How did you do it? Do you see any other patterns?

Each number in Pascal's triangle can be written as a combination using the notation ${}_nC_r$, where *n* is the number of objects in the set and *r* is the number selected. For example, you can express the **third row** as:

1
1 1
1 2 1 ⟹ ${}_2C_0$ ${}_2C_1$ ${}_2C_2$
1 3 3 1
1 4 6 4 1

Expand the following binomials by multiplying:

$(x+y)^2 =$

$(x+y)^3 =$

$(x+y)^4 =$

How do the **coefficients** of the simplified terms in your binomial expansions above relate to Pascal's triangle?

The coefficients in a binomial expansion can be determined from Pascal's triangle. In the expansion of $(x+y)^n$, where $n \in \mathbb{N}$, the coefficients of the terms are identical to the numbers in the $(n+1)^{th}$ row of Pascal's triangle.

Binomial	Pascal's Triangle in Binomial Expansion	Row
$(x+y)^0$	1	1
$(x+y)^1$	$1x + 1y$	2
$(x+y)^2$	$1x^2 + 2xy + 1y^2$	3
$(x+y)^3$	$1x^3 + 3x^2y + 3xy^2 + 1y^3$	4
$(x+y)^4$	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$	5

The coefficients in a binomial expansion can also be determined using combinations.

Pascal's Triangle	Combinations
1	${}_0C_0$
1 1	${}_1C_0$ ${}_1C_1$
1 2 1	${}_2C_0$ ${}_2C_1$ ${}_2C_2$
1 3 3 1	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$
1 4 6 4 1	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$
1 5 10 10 5 1	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$

$$
\begin{aligned}
{}_5C_2 &= \frac{5!}{3!2!} \\
&= \frac{(5)(4)}{2} \\
&= 10
\end{aligned}
$$

Note that ${}_5C_2$ represents the number of combinations of five items taken two at a time. In the expansion of $(x+y)^5$, it represents the coefficient of the term containing x^3y^2 and shows the number of selections possible for three x's and two y's.

Example 1

Expand Binomials

a) Expand $(p+q)^6$.

b) Identify patterns in the expansion of $(p+q)^6$.

$n=6$ From Pascal's △

$\boxed{1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1}$

Solution

a)

$$1p^6q^0 + 6p^5q^1 + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + 1p^0q^6$$

$$p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

Your Turn

a) What are the coefficients in the expansion of $(c+d)^5$?

b) Do you prefer to use Pascal's triangle or combinations to determine the coefficients in a binomial expansion? Why?

c) How many terms are in the expansion of $(c+d)^5$?

d) What is the simplified expression for the second term in the expansion of $(c+d)^5$ if the terms are written with descending powers of c?

You can use the **binomial theorem** to expand any power of a binomial expression.

$$(x+y)^n = {}_nC_0(x)^n(y)^0 + {}_nC_1(x)^{n-1}(y)^1 + {}_nC_2(x)^{n-2}(y)^2 + \cdots + {}_nC_{n-1}(x)^1(y)^{n-1} + {}_nC_n(x)^0(y)^n$$

binomial theorem

- used to expand $(x+y)^n$, $n \in \mathrm{N}$
- each term has the form ${}_nC_k(x)^{n-k}(y)^k$, where $k+1$ is the term number

The general term is given by:

$$t_{k+1} = {}_n\mathrm{C}_k(x)^{n-k}(y)^k$$

In this chapter, all binomial expansions will be written in descending order of the exponent of the first term in the binomial.

The following are some important observations about the expansion of $(x+y)^n$, where x and y represent the terms of the binomial and $n \in \mathrm{N}$:

- the expansion contains $n+1$ terms
- the number of objects, k, selected in the combination ${}_nC_k$ can be taken to match the number of factors of the second variable selected; that is, it is the same as the exponent on the second variable
- the general term, t_{k+1}, has the form

$${}_n\mathrm{C}_k(x)^{n-k}(y)^k$$

the same

- the sum of the exponents in any term of the expansion is n

<u>example:</u> $(2x-5y^3)^5$

From Pascal's triangle

1 5 10 10 5 1 as the leading coefficients.

$$\underline{1}(2x)^5(-5y^3)^0 + 5(2x)^4(-5y^3)^1 + 10(2x)^3(-5y^3)^2 +$$
$$+10(2x)^2(-5y^3)^3 + 5(2x)(-5y^3)^4 + \underline{1}(2x)^0(-5y^3)^5$$

$$32x^5 - 400x^4y^3 + 2000x^3y^6 - 5000x^2y^9$$
$$+6250xy^{12} - 3125y^{15}$$

11.3

3a, 4b, 5a

6 a, b

BLM 11–4 Section 11.3 Extra Practice

1. a) 1 2 1 **b)** 1 4 6 4 1 **c)** 1 6 15 20 15 6 1

2. a) ${}_4C_2$ **b)** ${}_7C_4$ **c)** ${}_{10}C_8$

3. a) 8 **b)** 10 **c)** $(n + 1)$

4. a) $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

b) $x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$

c) $64 + 48y + 12y^2 + y^3$

5. a) $27x^3 - 108x^2 + 144x - 64$

b) $16x^4 + 160x^3y + 600x^2y^2 + 1000xy^3 + 625y^4$

c) $a^5 - 10a^4b + 40a^3b^2 - 80a^2b^3 + 80ab^4 - 32b^5$

6. a) $35x^3y^4$ **b)** $34\ 992ab^7$ **c)** $-191\ 362\ 500x^5$

7. a) $(x - y)^5$ **b)** $(2 + x)^3$

8. $-\frac{135}{2}y^3$

9. 2160

10. 540