Chapter 11 – Permutations, Combinations, and the Binomial Theorem

11.1 Permutations

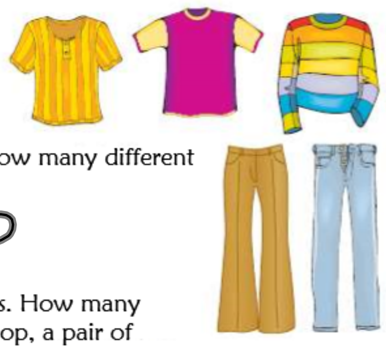
Counting methods are used to determine the number of members of a specific set as well as the outcomes of an event. You can display all of the possible choices using tables, lists, or tree diagrams and then count the number of outcomes. Another method of determining the number of possible outcomes is to use the **fundamental counting principle (FCP)**.

fundamental counting principle

- If one task can be performed in *a* ways and a second task can be performed in *b* ways, then the two tasks can be performed in $a \times b$ ways
- for example, a restaurant meal consists of one of two salad options, one of three entrees, and one of four desserts, so there are (2)(3)(4) or 24 possible meals

Example

You are packing clothing to go on a trip. You decide to take three different tops and two pairs of pants.



1. If all of the items go together, how many different outfits can you make?

6

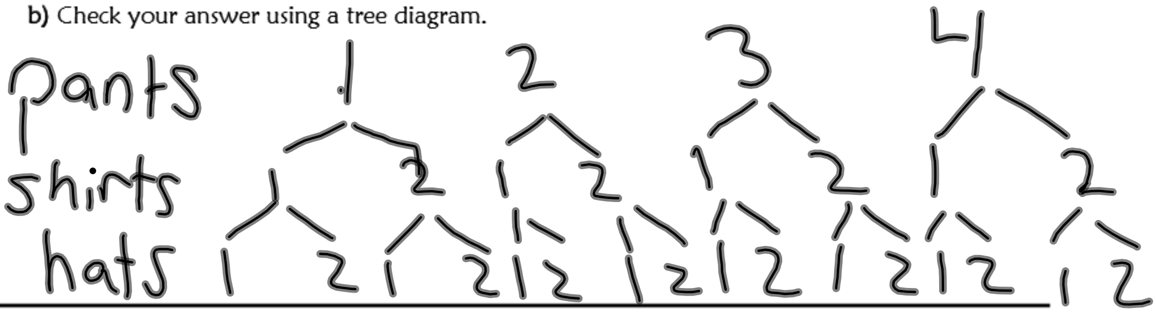
2. You also take two pairs of shoes. How many different outfits consisting of a top, a pair of pants, and a pair of shoes are possible?

12 $3 \times 2 \times 2 = 12$

3. a) Determine the number of different outfits you can make when you take four pairs of pants, two shirts, and two hats? (assume an outfit consists of a pair of pants, a shirt, and a hat).

$4 \times 2 \times 2 = 16$

b) Check your answer using a tree diagram.



Example 1

Arrangements With or Without Restrictions

a) A store manager has selected four possible applicants for two different positions at a department store. In how many ways can the manager fill the positions?

b) In how many ways can a teacher seat four girls and three boys in a row of seven seats if a boy must be seated at each end of the row?

Solution

a) Method 1: Use a Tree Diagram

position 1 1 2 3 4

position 2: 2 3 4 1 3 4 1 2 4 1 2 3

12 ways

Method 2: Use the Fundamental Counting Principle

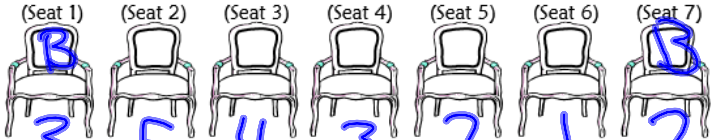
number for pos. 1 : 4

" " " 2 : 3

$4 \times 3 = 12$

b) Use seven chairs to represent the seven seats in the row.

(Seat 1) (Seat 2) (Seat 3) (Seat 4) (Seat 5) (Seat 6) (Seat 7)



B B

3 5 4 3 2 1 2 = 720

Your Turn

a) How many three-digit numbers can you make using the digits 1, 2, 3, 4, and 5 if repetition of the digits is not allowed?

$\underline{5}\ \underline{4}\ \underline{3} = 60$

b) How many three-digit numbers can you make using the digits 1, 2, 3, 4, and 5 if repetition of the digits is allowed?

$\underline{5}\ \underline{5}\ \underline{5} = 5^3 = 125$

Many examples involving arrangements require multiplying consecutive numbers decreasing to 1, such as 5x4x3x2x1. This product can be abbreviated as 5! and is read as "five **factorial**."

Thus, 5! = 5x4x3x2x1 = 120.

In general,

$n! = (n)(n - 1)(n - 2)\ldots(3)(2)(1)$, where $n \in N$.

factorial
- for any positive integer n, the product of all of the positive integers up to and including n
- $4! = (4)(3)(2)(1)$
- 0! is defined as 1

The arrangement of objects or people in a line is called a linear **permutation.** In a permutation, the order of the objects <u>IS</u> important. When the objects are distinguishable from one another, a new order of objects creates a new permutation.

For example, seven different objects can be arranged in 7! ways.

Thus, 7! = (7)(6)(5)(4)(3)(2)(1)

permutation
- an ordered arrangement or sequence of all or part of a set
- for example, the possible permutations of the letters A, B, and C are ABC, ACB, BAC, BCA, CAB, and CBA

$_nP_r$

The notation $_nP_r$ is used to represent the number of permutations, or arrangements in a definite order, of r items taken from a set of n distinct items. A formula for $_nP_r$ is $_nP_r = \frac{n!}{(n-r)!}$, $n \in N$.

Example 2a

Using Factorial Notation

If there are seven members on the student council, in how many ways can the council select three students to be the chair, the secretary, and the treasurer of the council?

Solution

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7\cdot6\cdot5\cdot\cancel{4!}}{\cancel{4!}} = 7\cdot6\cdot5$$
$$= 210$$

$\underline{7}\ \underline{6}\ \underline{5}$

Example 2b

Using Factorial Notation

a) Evaluate ${}_9P_4$ using factorial notation.
b) Show that 100! + 99! = 101(99!) without using technology.
c) Solve for n if ${}_nP_3 = 60$, where n is a natural number.

Solution

a) $_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9\cdot8\cdot7\cdot6\cdot\cancel{5!}}{\cancel{5!}} = 3024$

b) $100\cdot99! + 99! = 99!(100+1) = 101(99!)$

c) $_nP_3 = 60$ $\quad \frac{n!}{(n-3)!} = 60$ $\quad \frac{n(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 60$

$n^3 - 3n^2 + 2n = 60$

$n^3 - 3n^2 + 2n - 60$

1, 2, 3, 4, 5,

6, 10, 12, 15, 20, 30, 60

Your Turn

a) Evaluate ${}_7P_2$ using factorial notation. $= 42$
b) Show that 5! – 3! = 19(3!).
c) Solve for n if ${}_nP_2 = 56$. $n = 8$

Solution

b) $5\cdot4\cdot3! - 3!$
$= 20\cdot3! - 3!$
$= 19\cdot3!$

c) $\frac{n!}{(n-2)!} = 56$

$\frac{n\cdot(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 56$

$n(n-1) = 56$

$n^2 - n = 56$

$n^2 - n - 56 = 0$

$(n-8)(n+7) = 0$

$\boxed{n=8}$ $\boxed{\cancel{n=-7}}$